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An Analytical Solution for the Nonlinear Frequency Response of Radiant Heat Transfer

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An analytical solution is presented for the nonlinear frequency response of a system in which radiant heat transfer is coupled with conduction. The model is that of an object inside an evacuated enclosure whose wall temperature oscillates about a mean. A perturbation analysis solves the describing equations for oscillatory amplitudes less than one-tenth the mean wall temperature. An exact solution to a Ricatti differential equation verifies the perturbation analysis. These solutions show that, contrary to linear experience, the average object temperature over an oscillatory cycle exceeds the mean at the wall.

The complete solution for the frequency response can be divided into three regimes. In the first, at low frequencies, all response is truly linear. In the second, higher harmonics become important, but the response remains that of a lumped parameter system. In the third, response changes with distance from the object surface and the response at the surface dominates. Nonlinearities are important in the second and third regimes. The solution has several applications.

In many processes thermal radiation forms a significant part of the total energy transfer. Yet the response of thermal radiation to temperature changes is not well understood because the equations describing radiant heat transfer are fundamentally nonlinear. This paper analyzes the nonlinear response of radiant heat transfer to temperature oscillations, and describes the response in detail. Some unusual and unexpected results are obtained.

Radiant heat transfer is particularly important in furnaces, high-temperature processes, and in evacuated or low-pressure systems such as outer space. In general, wherever the temperature is high so that the radiant heat flux is substantial, or the pressure is low so that convection is negligible, radiant heat transfer will be significant. Anywhere radiant heat transfer is important and the temperature changes, the dynamic response will be of interest.

A consequence of quantum statistics is that the heat flux in radiant transfer varies as the fourth power of the absolute temperature (*11*). Thus in systems where radiant heat transfer is important, the differential equations describing temperature changes are nonlinear. In general, because of the dominant fourth-power nonlinearity, these differential equations cannot be solved in closed form.

In radiant heat transfer systems where the effects of convection and conduction are both negligible, the dynamic response is described by a first-order ordinary differential equation containing a nonlinear term. This equation can be solved exactly for a few simple cases, and numerical solutions can be obtained by conventional methods.

When conduction as well as radiation becomes important, however, the system is described by the heat conduction equation with a linear flux boundary condition on one surface and a nonlinear flux boundary condition on the other. No complete exact solutions are known for the heat conduction equation when one of its boundary conditions is nonlinear and time varying and numerical solutions are generally difficult to obtain. Even when numerical solutions can be achieved, the information they contain is highly restricted. It is difficult to extrapolate the solutions with confidence to slightly different systems, and the physical insight one attains is usually minimal.

A number of approximate solutions to problems involving both conduction and radiation have been obtained (*1, 3, 7, 10*). These solutions are in general quite complex and restricted to particular problems. They thus do not afford the broad intuitive understanding of the interaction of radiant and conductive heat transfer that is required for engineering analysis and design.

Faced with a similar inadequacy of particular numerical solutions or complex approximations, one often characterizes a linear system by its steady state response to a forced sinusoidal input. The frequency response, as this characterization is called, furnishes a "fingerprint" of the system, and affords a basis from which the response to different inputs may be determined. Since radiant heat transfer is nonlinear, and the superposition principle does not hold, the frequency response is not so valuable quantitatively as it is for linear systems. Nevertheless, the frequency response of radiant heat transfer does yield considerable qualitative information and intuitive understanding.

PHYSICAL MODEL AND ASSUMPTIONS

The physical model, as shown in Figure 1, consists of a homogeneous slab in an evacuated enclosure where the wall temperature oscillates around a mean. The slab is an infinite sheet of finite thickness, parallel to the walls of the enclosure, with the same sinusoidal oscillation of temperature impinging on both surfaces. The other assumptions involved are: (1) Convective heat transfer between the walls and the slab is completely negligible. (2) Heat transfer within the slab occurs only by conduction; that is, the slab is optically opaque. (3) Physical properties of the slab are constant.

THE HEAT TRANSFER EQUATION

The Equation and Boundary Conditions

Based on the physical model and the assumptions given above, the heat transfer equation and its boundary conditions may be expressed as follows:

Within the slab, an energy balance yields the heat conduction equation

$$k \frac{\partial^2 T'}{\partial z^2} = \rho C_p \frac{\partial T'}{\partial \tau} \qquad (1)$$

At the slab surface, heat is exchanged with the enclosure walls by radiation, and with the interior of the slab by conduction. Since there is no accumulation of heat at the surface, the conductive heat flux must equal that from radiation. Thus

$$k A \frac{\partial T'}{\partial z} = F \sigma A (T_w'^4 - T'^4), z = b \qquad (2)$$

where

$$T_w' = T_M + (\delta T_w) \sin \omega\tau \qquad (3)$$

At the center of the slab, symmetry imposes the boundary condition

$$\frac{\partial T'}{\partial z} = 0, \quad z = 0 \qquad (4)$$

Since only the steady state solution is desired, the initial condition within the slab is inconsequential. The same boundary conditions would hold for a slab insulated at $z = 0$ and exposed to temperature oscillations on only one surface.

In the dimensionless variables

$$\text{(i)}\ x = z/b \qquad \text{(iii)}\ T = T'/T_M$$
$$\text{(ii)}\ t = \omega\tau \qquad \text{(iv)}\ a = \delta T_w/T_M \qquad (5)$$

the equation becomes

$$\frac{\partial T}{\partial t} = B \frac{\partial^2 T}{\partial x^2} \qquad (6)$$

with boundary conditions

$$\frac{\partial T}{\partial x} = 0, \quad x = 0 \qquad (7)$$

$$\frac{\partial T}{\partial x} = G[T_w^4 - T^4], \quad x = 1 \qquad (8)$$

where

$$T_w = 1 + a \sin t \qquad (9)$$

Dimensionless Groups

The two dimensionless groups characterizing the system are

$$B = \frac{k}{\rho C_p \omega b^2} \text{ and } G = \frac{F \sigma b T_M^3}{k}$$

The group G is the ratio of radiant heat transfer to that by conduction. The group B is a modified Fourier number (*8*). But in terms of the frequency response of thermal radiation it has yet another, more intuitive meaning. The group $k/\rho C_p$ is the thermal diffusivity, or the diffusion coefficient for heat within the slab. The group $\left(\frac{k}{\rho C_p} \Big/ b \right)$, then, in a random walk sense is an average "velocity" of heat conduction. When this velocity is divided by the frequency of oscillation at the wall, we obtain an effective "wavelength" for the waves of heat passing into the slab. The overall group B, therefore, can be considered as the ratio of this heat transfer wavelength to the thickness of the slab.

When the ratio B is large, that is, when the frequency or the slab thickness is small or the thermal conductivity is large, the wavelength is much larger than the slab thickness. In this case, every part of the slab sees much the same part of the thermal wave, and intuitively the entire slab would be expected to act more or less as a unit, as if its properties were lumped together at a point. On the other hand, when this ratio is small, that is, when the frequency or the slab thickness is large or the thermal conductivity is small, the wavelength is much less than the slab thickness. Thus different parts of the slab see very different parts of the thermal waves, and the responses at various slab depths would be assumed to differ widely.

The Problem

The problem is to solve the heat conduction equation for a slab with an oscillating nonlinear flux boundary condition prescribed along one surface. The same problem with T_w constant (that is, for a step change in the surrounding wall temperature) has been solved by Abarbanel (*1*). His method involves the interative solution of a nonlinear integral equation for the surface temperature and the convolution of the fourth power of the surface temperature with an exponential to yield the profile inside the slab. This complicated method is not particularly attractive for problems having an oscillating boundary condition.

For a related problem with oscillating T_w, Olmstead and Raynor (*10*) have obtained solutions by linearizing the T^4 term in the surface boundary condition. But no check is offered on the solutions so obtained, and the question remains as to what extent linearization accounts for the effects of so dominant a nonlinearity. What is needed is an analytical method that retains the simplicity of linearization and yet offers a means of determining the extent to which linearization is valid.

PERTURBATION TECHNIQUE

The amplitude a of the wall temperature oscillation must physically be less than one, and in practice will be small. Thus, it seems reasonable to regard the fluctuating

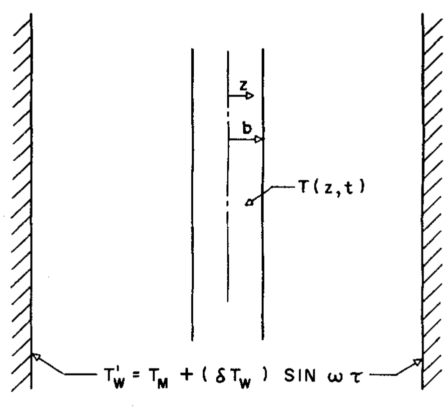

Fig. 1. Physical model.

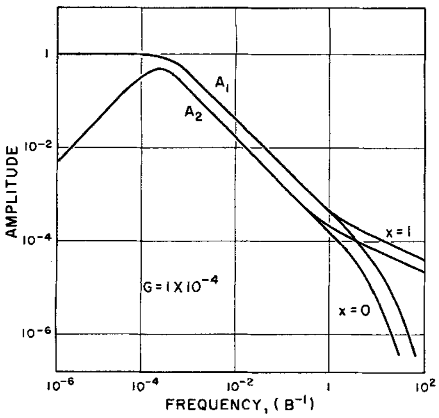

Fig. 2. First and second harmonic amplitudes for low ratio of radiant to conductive heat transfer.

part of T_w as a perturbation, and to expand the temperature T as a power series in a, that is

$$T(x,t) = T_0(x,t) + a\,T_1(x,t) + a^2\,T_2(x,t) + \ldots.. \quad (10)$$

This technique will remove the obstacle posed by the nonlinear boundary condition. Equation (10) is substituted into Equations (6) through (8), the coefficients of powers of a are set individually equal to zero, and the resulting set of equations is solved for the T_n. A coupled hierarchy of heat conduction equations with linear boundary conditions is obtained. These equations can be solved successively by conventional transform methods (*4*) to give the T_n, and thus the solution for the slab temperature.

This perturbation power series technique is one of a variety of perturbation techniques that are useful for analyzing nonlinear equations (*2*). In this instance, because thermal conduction is inherently a dissipative process, the temperature is well behaved and the perturbation series (10) will converge. If a is very small, or conditions are such that the higher T_n become negligible, the series will converge very rapidly and the first few terms will suffice to give an accurate solution.

The first term in (10), T_0, is the solution for no oscillation. When there is oscillation, the second term in the series, $T_1(x,t)$, is the linearized solution for the fluctuating component of the slab temperature. The extent to which linearization is valid is indicated by the relative contribution of the higher terms in the series, particularly the leading "nonlinear" term T_2. With the perturbation technique one can solve for $T_2(x,t)$, and thus determine the importance of nonlinearities in the dynamic response of radiant heat transfer.

Perturbation Solution

The perturbation technique yields the equations

$$\frac{\partial T_n}{\partial t} = B\,\frac{\partial^2 T_n}{\partial x^2} \quad (11)$$

with the interior boundary conditions

$$\frac{\partial T_n}{\partial x} = 0, \quad x = 0 \quad (12)$$

for $n = 0$, 1, and 2. The exterior boundary conditions at the surface $x = 1$ are, respectively

$$\frac{\partial T_0}{\partial x} = G(1 - T_0^4) \quad (13)$$

$$\frac{\partial T_1}{\partial x} = G(4 \sin t - 4\,T_0^3\,T_1) \quad (14)$$

and

$$\frac{\partial T_2}{\partial x} = G(6 \sin^2 t - 4T_0^3\,T_2 - 6T_0^2\,T_1^2) \quad (15)$$

Only the steady state solutions for each of the T_n are desired, since the frequency response is a steady state characterization and

$$T(x,t)_{ss} = T_{0_{ss}} + a\,T_1(x,t)_{ss} + a^2\,T_2(x,t)_{ss} + \ldots$$

In the steady state with no oscillations, one can easily show that

$$T_0(x,t)_{ss} \equiv 1 \quad (16)$$

First Harmonic

To solve Equation (11) with boundary conditions (12) and (14) for $T_1(x,t)$, the Laplace transform is applied with respect to t (*4*). The transient terms are neglected the steady state solution is obtained.

$$T_1(x,t) = 4G\,\gamma^{-1}(y)\,[\alpha(x,y)\sin t + \beta(x,y)\cos t] \quad (17)$$

where

$$y \equiv \frac{1}{\sqrt{2B}} \quad (18a)$$

$$\alpha(x,y) \equiv Q(y)\,C(x,y) + R(y)\,S(x,y) \quad (18b)$$

$$\beta(x,y) \equiv Q(y)\,S(x,y) - R(y)\,C(x,y) \quad (18c)$$

$$\gamma(y) \equiv Q^2(y) + R^2(y) \quad (18d)$$

$$Q(y) \equiv y \sinh y \cos y - y \sin y \cosh y + 4G \cosh y \cos y \quad (18e)$$

$$R(y) \equiv y \sinh y \cos y + y \sin y \cosh y + 4G \sinh y \sin y \quad (18f)$$

$$C(x,y) \equiv \cosh yx \cos yx \quad (18g)$$

$$S(x,y) \equiv \sinh yx \sin yx \quad (18h)$$

Second Harmonic

To solve the equation for $T_2(x,t)$, again the Laplace transform is used. Substituting in the solutions for T_0 and T_1, and again neglecting the transient terms in the inverse transform, one obtains the steady state solution for T_2.

$$T_2(x,t) = \mu_2 + 3G\,\gamma^{-1}(Y)\,[\Delta(x,Y)\sin 2t + \eta(x,Y)\cos 2t] \quad (19)$$

where

$$Y \equiv \frac{1}{\sqrt{B}} = y\sqrt{2} \quad (20a)$$

$$\Delta(x,Y) \equiv (C_4)\,\alpha(x,Y) - (C_3)\,\beta(x,Y) \quad (20b)$$

$$\eta(x,Y) \equiv (C_3)\,\alpha(x,Y) + (C_4)\,\beta(x,Y) \quad (20c)$$

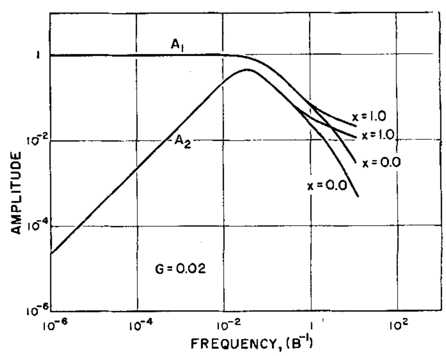

Fig. 3. First and second harmonic amplitudes for moderate ratio of radiant to conductive heat transfer.

$$\mu_2 \equiv \frac{3}{4} \gamma^{-2}(y) \, [\gamma^2(y) - 16G^2(\alpha^2(1,y) + \beta^2(1,y))] \quad (20d)$$

$$C_3 \equiv -1 + 16G^2 \gamma^{-2}(y) \, [\alpha^2(1,y) - \beta^2(1,y)] \quad (20e)$$

$$C_4 \equiv 32G^2 \alpha(1,y) \, \beta(1,y) \, \gamma^{-2}(y) \quad (20f)$$

Complete Solution

Now the complete solution may be rewritten in the form

$$T(x,t) = 1 + a[A_1(x) \sin (t + \phi_1(x))] + a^2 [\mu_2 + A_2(x) \sin (2t + \phi_2(x))] + \ldots \quad (21)$$

where, from (17)

$$A_1(x) = 4G \gamma^{-1}(y) [\alpha^2(x,y) + \beta^2(x,y)]^{1/2} \quad (22a)$$

$$\phi_1(x) = \tan^{-1} [\beta(x,y)/\alpha(x,y)] \quad (22b)$$

and, from (19)

$$A_2(x) = 3G \gamma^{-1}(Y) \, [\Delta^2(x,Y) + \eta^2(x,Y)]^{1/2} \quad (23a)$$

$$\phi_2(x) = \tan^{-1} [\eta(x,Y)/\Delta(x,Y)] \quad (23b)$$

$A_1(x)$ and $A_2(x)$ are called the amplitudes of the first and second harmonics, respectively. The functions $\phi_1(x)$ and $\phi_2(x)$ are the corresponding phase angles. Taking the time average of (21) one obtains

$$\langle T \rangle = 1 + a^2 \mu_2 + O(a^4) \quad (24)$$

since the time average of a pure harmonic is zero. Thus at every position in the slab, the time-average temperature over an oscillatory cycle exceeds the mean at the wall. This unexpected difference between the slab and wall mean temperatures is due to the nonlinearity in radiant heat transfer.

SOLUTION DESCRIPTION

There exists no completely satisfactory representation for nonlinear frequency response. In the strict sense of the word, one cannot determine a "transfer function" for a nonlinear system because the transfer function is defined in terms of the linear Laplace transform. The "describing function" representation often used for nonlinear systems (*6*) depicts only the first harmonic. However, one can describe the radiant heat transfer system—at least to the second order in the oscillatory amplitude *a*—in terms of the five functions that make up the solution (21). Linear frequency response characterizes a system by the behavior of the response amplitude and phase angle. For thermal radiation the behavior of two amplitudes, two phase angles, and the steady state time average is depicted.

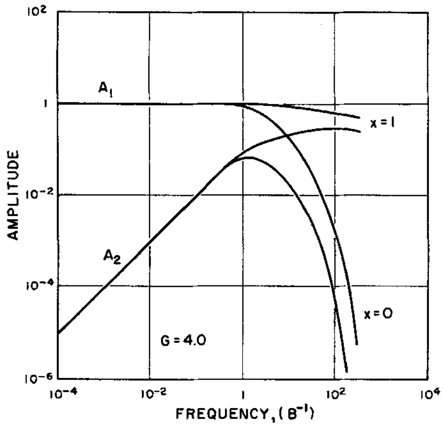

Fig. 4. First and second harmonic amplitudes for high ratio of radiant to conductive heat transfer.

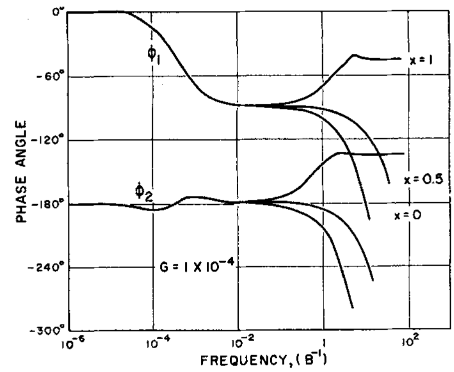

Fig. 5. First and second harmonic phase angles for low ratio of radiant to conductive heat transfer.

Amplitudes

In Figures 2, 3, and 4, the logarithms of the amplitudes of the first and second harmonics A_1 and A_2 are graphed against the log of the dimensionless frequency $Y^2 = B^{-1}$ at three different values of G. At all values of G, for small Y^2, or large B, the response is the same at all points within the slab. This lumped response confirms the intuitive expectations. Within this lumped parameter region the amplitude of the first harmonic

$$A_1 = \frac{4}{\sqrt{M^2 + 16}}, \quad \text{where } M \equiv Y^2/G$$

The amplitude is close to one from $Y^2 = 0$ to the neighborhood of $Y^2 = 4G$. Somewhat above $Y^2 = 4G$, or $M = 4$, $A_1 = 4M^{-1}$, which yields the straight line with slope -1 in Figure 2. The point where the two straight lines $A_1 = 1$ and $A_1 = 4M^{-1}$ would meet if extrapolated, $M = 4$, is called the corner frequency.

Above a particular frequency, the responses at different positions within the slab diverge, again confirming intuition. For the first harmonic amplitude, the frequency at which slab position becomes important varies slightly with G. For $G = 1 \times 10^{-4}$, the transition from lumped to distributed solution occurs roughly at $Y^2 = ½$. For $G = 4.0$, the transition occurs more nearly at $Y^2 = ¼$. For values of Y above the transition

$$A_1(x) \approx 8G \, Y^{-1} \exp \left[-\frac{Y}{\sqrt{2}} (1-x) \right]$$

The amplitude at the surface, $x = 1.0$, is $8G\,Y^{-1}$, a straight line with slope $-½$ on the graphs. At positions within the slab, the amplitude falls off exponentially.

With the important exception that $A_2 \to 0$ at zero frequency, the behavior of the second harmonic amplitude A_2 parallels the behavior of A_1. At low frequencies, A_2 is a rising line, $A_2 = ⅜ \, M$. Near $M = 2$, this line bends over. It then passes through a maximum at $M = 2$, where the amplitude $A_2 = 0.43732$, and becomes a falling line $A_2 = 3/2 \, M^{-1}$. This falling line parallels the declining lumped parameter amplitude of the first harmonic. And at a transition frequency (roughly $Y^2 = ¼$) the second harmonic amplitude, too, differs with position in the slab. For very large values of Y the asymptotic distributed form of A_2 becomes

$$A_2(x) \approx 3\sqrt{2} \, G \, Y^{-1} \exp \, [-Y(1-x)]$$

Again the amplitude at the slab surface falls off with the square root of the frequency, while inside the slab it declines exponentially.

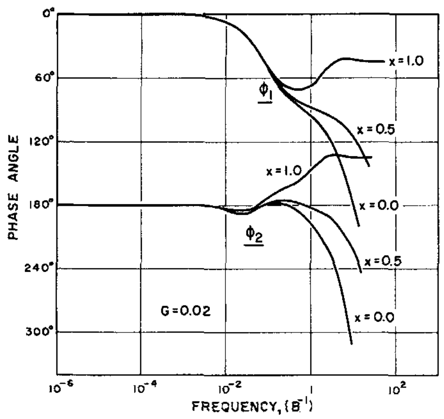

Fig. 6. First and second harmonic phase angles for moderate ratio of radiant to conductive heat transfer.

Phase Angles

In Figures 5, 6, and 7, the first and second harmonic phase angles ϕ_1 and ϕ_2 are graphed against the log of Y^2 for three values of G. Like the amplitudes, the phase angles behave at low frequencies as if the slab properties were lumped together. As the frequency rises, they increasingly show the effects of parameter distribution. In the lumped parameter region

$$\phi_1 = \tan^{-1}\left(-\frac{1}{4}M\right)$$

The transition to the region where slab position is important occurs roughly at $Y^2 = 1/40$, much earlier than the corresponding transition for the amplitudes. In the distributed parameter region, at the surface of the slab, ϕ_1 increases, then levels asymptotically to —45 deg.; ϕ_2 also increases and levels out to —135 deg. Inside the slab, the phase angles for both harmonics decrease exponentially.

Time-Average Term

In Figure 8 the log of the time-average term μ_2 is graphed against the log of M. For small G, μ_2 behaves as if the properties of the slab were lumped together at a point. At larger values of G, the distributed character of the slab becomes important, and the distributed and lumped parameter solutions diverge, coming together only in their common limit of 3/4 at high frequencies. On the rising line with slope +2, for small M

$$\mu_2 \approx \frac{3}{4}\left(\frac{M^2}{16}\right)\left[1 + \frac{8}{3}G\right]$$

At higher frequencies, where the curve bends over

$$\mu_2 \approx \frac{3}{4}\left[1 - \frac{32\,G}{M}\right]$$

SOLUTION DISCUSSION

Thus from an engineering standpoint, the frequency response of radiant heat transfer can be divided into three separate regimes.

Linear Regime

The first is the linear regime, which predominates through the low frequencies. It terminates at the first transition to either of the other two regimes. Within the linear regime, thermal radiation is truly a linear process. The amplitude of the first harmonic is almost identically one, and the second harmonic amplitude and μ_2 are both quite negligible. In this regime, all effects arising from the nonlinearity of thermal radiation or the thickness of the slab may legitimately be neglected. In other words, all responses may be calculated assuming that the heat transfer is linear and that the slab properties are lumped together at a point.

Harmonic Regime

The transition from the linear regime to the second regime begins roughly at $M = 0.1$, or a transition frequency

$$\omega_2 = \frac{0.1\,\sigma\,F\,T_M^3}{\rho\,b\,C_p}$$

The second regime ends where distributed parameter effects become important. This second regime is called the "harmonic" regime, because here the second harmonic contributes significantly to the frequency response. In the harmonic regime the amplitude A_2 rises to a maximum of 0.437 at a frequency where $A_1 = 0.895$. Thereafter the ratio of A_2 to A_1 becomes a constant 3/8. For small G, the time-average μ_2 attains its maximum value of 3/4 well within the harmonic regime. Since the ratio of response contributions from the second and first harmonics is, from (21)

$$R_{21} = \left[\frac{\mu_2 + A_2 \sin\,(2t + \phi_2)}{A_1 \sin\,(t + \phi_1)}\right] a$$

it is apparent that even the minimum value of $R_{21} = (\mu_2 - A_2)\,a/A_1$ becomes large at modest frequencies. For example, with $G = 1 \times 10^{-4}$ and $Y^2 = 4 \times 10^{-2}$, even with $a = 0.04$, the second harmonic contribution is three times as great as that from the first harmonic (which is the linearized solution). Thus nonlinear effects become extremely important in the harmonic regime, although one can still assume here that slab properties are lumped together at a point.

Surface-Dominated Regime

The lumped parameter behavior breaks down at sufficiently high frequencies, and one enters the third regime. This distributed parameter region is called the "surface-dominated" regime because the response at the surface becomes very much greater than the response within the slab. The transition to the surface-dominated regime occurs at $Y^2 = 1/40$ for phase angles and at $Y^2 = 1/4$ for amplitudes. The latter corresponds to a transition frequency

$$\omega_3 = \frac{k}{4\rho\,C_p\,b^2}$$

In the surface-dominated regime, μ_2 attains its maximum value of 3/4, while the amplitudes of oscillation, even at the surface, decline with increasing frequency. Thus in this regime nonlinearities become all important, and the linearized solution fails to account for the dynamic response of radiant heat transfer.

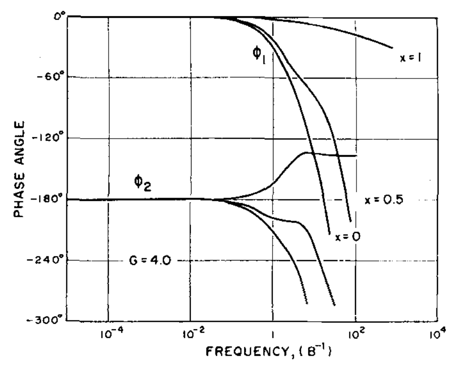

Fig. 7. First and second harmonic phase angles for high ratio of radiant to conductive heat transfer.

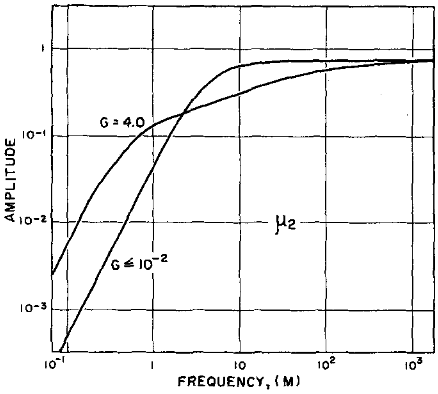

Fig. 8. Second harmonic time-average amplitude.

For small G (for example, $G = 1 \times 10^{-4}$) all three regimes are obtained. As G increases, the two transitions occur in the same frequency range and the harmonic regime is small. As G becomes larger still (for example, $G = 4.0$) the transition to surface domination occurs first and the harmonic regime never appears. Systems with small values of G have low mean temperatures, as in cryogenic systems, or contain relatively thin slabs with the large thermal conductivities typical of metals. Systems with high mean temperatures, thick slabs, or low thermal conductivities usually have large values of G.

PERTURBATION VALIDITY

Since physical conclusions have been drawn from the perturbation solution (21), particularly from the second harmonic, it seems worthwhile to check the solution's validity. First the perturbation technique itself must be checked by testing it on a related nonlinear equation that can be solved exactly. Then, if it is found that the technique accounts correctly for nonlinearities, the extent to which the exact solution is well represented by the first two harmonics must be determined.

Exact Analog Solution

When the slab acts as if its properties were lumped together at a point (that is, in the linear and harmonic regimes), the frequency response is described by the ordinary differential equation

$$M\frac{dT}{dt} + T^4 = (1 + a\sin t)^4 \tag{25}$$

No exact solutions are known for this equation, but its second-power analog, the Ricatti equation

$$M\frac{dT}{dt} + T^2 = (1 + a\sin t)^2 \tag{26}$$

can be solved exactly (5, 9). Equation (26) can be considered as describing the nonlinear frequency response of some lumped parameter system in which heat transfer follows a (hypothetical) T^2 law. For the nonlinear analog equation (26), the perturbation technique predicts

$$<T> = 1 + a^2\left[\frac{1}{4}\frac{M^2}{M^2+4}\right] + 0(a^4) \tag{27}$$

to be the steady state time average. This prediction can be checked by determining $<T>$ exactly.

The substitution $T = (M/u)\, du/dt$ transforms (26) into the second-order linear equation (5)

$$\frac{d^2u}{dt^2} - \frac{1}{M^2}(1 + a\sin t)^2 u = 0 \tag{28}$$

Equation (28) is a form of Hill's equation, of which Mathieu's equation is another particular example. Techniques for solving Hill's equation are described in detail by Whittaker and Watson (12).

From Floquet theory it is known that the general periodic solutions to (28) have the form

$$u(t') = e^{\mu t'}G(t') \tag{29}$$

where $t' = \frac{1}{2}t$, and $G(t')$ is periodic in 2π. Consider the time average of T in the steady state. Solutions of the form (29) are the periodic or steady state solutions. Thus

$$<T> = <M\frac{1}{u}\frac{du}{dt}> = M<\frac{d\ln u}{dt}>$$

$$= \frac{M}{2\pi}[\ln u(2\pi) - \ln u(0)] \tag{30a}$$

which yields

$$<T> = \frac{M}{2}\mu + \frac{M}{2\pi}[\ln G(2\pi) - \ln G(0)] = \frac{M\mu}{2} \tag{30b}$$

Whittaker and Watson show that μ is the root of the transcendental equation

$$\sin^2\left(\frac{1}{2}\pi i \mu\right) = \Delta(0)\sin^2\left(\frac{1}{2}\pi\sqrt{\theta_o}\right) \tag{31}$$

where

$$\theta_o = -\frac{4}{M^2}(1 + \underline{a}^2/2) \tag{32}$$

and $\Delta(0)$ is the Hill infinite determinant (12).

To find μ, the determinant $\Delta(0)$ must be evaluated. Since the determinant is infinite, its expansion yields an infinite number of terms, each of which contains infinite series. Fortunately, each succeeding term is smaller than the one preceding it by a factor of a^2. Thus one can terminate the expansion with the a^2 term, if a is small, and obtain an excellent approximation to the value of the total determinant. To order a^2, the determinant expansion yields

$$\Delta(0) = 1 - a^2\left[\frac{2\pi}{M(M^2+4)}\coth\left(\frac{\pi}{M}\sqrt{1 + a^2/2}\right)\right] + 0(a^4) \tag{33}$$

One substitutes this expression for $\Delta(0)$ into (31) and solves for μ. Using standard approximations for small a, one then has, after some rearrangement

$$\mu = \frac{2}{\pi}\left[\frac{\pi}{M}\sqrt{1 + a^2/2} - \frac{a^2\pi}{M(M^2+4)}\right] + 0(a^4)$$

or

$$\mu = \frac{2}{M}\left[1 + \frac{a^2M^2/4}{M^2+4}\right] + 0(a^4) \tag{34}$$

which, with (30b), exactly confirms the perturbation solution for $<T>$ given in Equation (27). Thus the perturbation technique appears to account correctly for nonlinearities of this kind, and it may be used with confidence.

Solution Accuracy

The solution obtained in (21) is accurate only for small values of the oscillatory amplitude a. It is difficult to assign an upper limit to acceptable values for a. However, computer solutions to Equation (25) obtained for values of a as large as 0.17 agree with the analytical solution

(21) in the lumped parameter region to within a few percent. Judging from the agreement of the computed solutions and from the degree of approximation involved in obtaining the exact Ricatti solution (34), one can say that the perturbation solution (21) will certainly be valid for $a \leqq 0.1$, and will probably hold with fair accuracy for a as large as 0.2. For larger a, the contributions of the third and higher harmonics will become important.

APPLICATIONS

Several potential applications stem from this analysis. One is the prediction of thermal radiation's response to other forms of temperature change. The analysis indicates that low frequency changes will induce linear responses. For most engineering systems, the response will be linear if the frequency is less than 10^{-4} sec.$^{-1}$, or roughly one cycle every three hours.

When the system is in the surface-dominated regime, virtually all the temperature change occurs in a thin surface layer. Thus one has essentially a boundary-layer phenomenon to which some of the techniques of boundary-layer theory may be applied. The integral methods reviewed by Goodman (7) fit into this category. They can be applied whenever $\omega > \omega_3$. For a copper slab three inches thick, they can be applied whenever the temperature changes faster than a cycle per minute. For a refractory slab the same width, the required rate is only one cycle every few hours.

The perturbation technique can also be applied to spheres and cylinders, as well as slabs. The frequency responses for these geometries are much the same as the slab response described above. For any of these geometries, the nonlinear frequency response can be applied to the design and control of systems involving radiant heat transfer coupled with conduction.

SUMMARY AND CONCLUSIONS

The nonlinear frequency response for a system in which radiant heat transfer is coupled with conduction has been presented. The complete frequency response of thermal radiation for small oscillations is presented analytically and in a series of graphs. The perturbation solutions obtained hold for a variety of system conditions, and are checked by exact analytical solutions to analogous equations, and by numerical computation. A number of intuitive estimates are confirmed by the solutions, which also predict several unusual results.

Physically, one may summarize the solution as follows:

(1) For relative changes in temperature less than one-tenth and for frequencies less than ω_2 and ω_3, radiant heat transfer may be assumed linear.

(2) For frequencies less than ω_3 the thermal wavelength is sufficiently large compared to body dimensions that the entire body responds as a single unit.

(3) For frequencies greater than ω_2 or ω_3 the nonlinearity inherent in thermal radiation becomes important and radiant heat transfer cannot be treated linearly.

(4) For frequencies greater than ω_3 the thermal wavelength becomes small compared to body dimensions. The temperature wave thus barely penetrates the slab in each cycle, and the only significant response occurs at or near the surface.

(5) For all frequencies, the average temperature in the slab is greater than the mean temperature of the wall.

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NOTATION

A	= surface area involved in radiant heat transfer
A_1	= amplitude of the first harmonic
A_2	= amplitude of the second harmonic
a	= dimensionless amplitude of the wall temperature oscillation = $\delta T_w / T_M$
B	= characteristic dimensionless group = $k/\rho C_p \omega b^2$
b	= slab half-thickness
C_p	= heat capacity of the object
F	= overall interchange factor for radiant heat transfer. For an infinite slab parallel to an infinite wall, $F = (\epsilon_s^{-1} + \epsilon_w^{-1} - 1)^{-1}$, where the ϵ's are the respective emissivities.
G	= characteristic dimensionless group = $\sigma F b T_M^3 / k$
k	= thermal conductivity of the object
M	= characteristic dimensionless group = $\rho V C_p \omega / \sigma F A T_M^3$
T	= dimensionless slab temperature = T'/T_M
T'	= dimensional slab temperature
T_M	= mean wall temperature
T_w'	= temperature of the wall = $T_M + \delta T_w \sin \omega\tau = T_M(1 + a \sin t)$
$<T>$	= steady state time-average of T
t	= dimensionless time = $\omega\tau$
x	= dimensionless distance = z/b
Y	= characteristic dimensionless group = $B^{-1/2}$
y	= characteristic dimensionless group = $(2B)^{-1/2}$
z	= dimensional distance

Greek Letters

μ_2	= second harmonic time-average term
ρ	= density of the slab
σ	= Stefan-Boltzmann radiation constant
τ	= dimensional time
ϕ_1	= first harmonic phase angle
ϕ_2	= second harmonic phase angle
ω	= frequency of the wall temperature oscillation
ω_2	= frequency of transition to the harmonic regime = $\sigma F T_M^3 / 10\ b C_p$
ω_3	= frequency of transition to the surface-dominated regime = $k/4\ C_p b^2$

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